Dirichlet Problem

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1 Overview

Today we want to solve the Dirichlet problem. That is, let Ω be a bounded C^2 domain in $\mathbb{R}^n$, $L = a^{ij} D_{ij} + b^i D_i + c$ be an elliptic operator with coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$ and $c \leq 0$ in Ω. For every $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$, we want to solve for $u \in C^0(\overline{\Omega}) \cap C^{2,\mu}(\Omega)$ such that

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

We will solve (1) in the following steps:

1. Solve the Dirichlet problem for the Laplace operator $\Delta = D_{11} + D_{22} + \cdots + D_{nn}$ on a ball B, i.e. solve

$$\begin{aligned} \Delta u &= f \text{ in } B, \\ u &= \varphi \text{ on } \partial B. \end{aligned}$$

   We will mostly handle this later when discussing equations in divergence form.

2. Show that Step 1 implies that we can solve the Dirichlet problem for L on a ball B, i.e. solve

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega. \end{aligned}$$

3. Use the Perron method of subsolutions and supersolutions, which assumes Step 2, to construct a solution $u \in C^{2,\mu}(\Omega)$ to $Lu = f$ in Ω.

4. Use barriers to show that the solution constructed in Step 3 satisfies $u \in C^0(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$.

Recall that we have the following theorems and estimates in this situation:

1. By the maximum principle, the solution to the Dirichlet problem (1) is unique.

2. By the maximum principle, if $v, w \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that $Lv \geq Lw$ in Ω and $v \leq w$ on $\partial\Omega$, then $v \leq w$ in Ω.

3. A priori estimate: if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ solves the Dirichlet problem (1), then
$$|u|_{0;\Omega} \leq |\varphi|_{0;\partial\Omega} + C|f|_{0;\Omega}$$
for some constant $C = C(n, L, \Omega) \in (0, \infty)$.
4. Interior Schauder estimate: if $u \in C^{2,\mu}(\Omega)$ solves $Lu = f$ in Ω, then
$$|u|_{2,\mu;\Omega'} \leq C(|u|_{0;\Omega} + |f|_{0,\mu;\Omega})$$
for all $\Omega' \subset\subset \Omega$ for some constant $C = C(n, L, \Omega', \Omega) \in (0, \infty)$.
5. Global Schauder estimate: if $u \in C^{2,\mu}(\overline{\Omega})$ solves the Dirichlet problem (1) and $\varphi \in C^{2,\mu}(\overline{\Omega})$, then
$$|u|_{2,\mu;\Omega} \leq C(|f|_{0,\mu;\Omega} + |\varphi|_{2,\mu;\Omega})$$
for some constant $C = C(n, L, \Omega) \in (0, \infty)$.

2 Step 1: Dirichlet problem for the Laplacian

Let B be a ball in $\mathbb{R}^n$. I claim that for every $f, \varphi \in C^\infty(\overline{B})$ there exists a solution $u \in C^\infty(\overline{B})$ to the Dirichlet problem
$$\begin{aligned} \Delta u &= f \text{ in } B, \\ u &= \varphi \text{ on } \partial B. \end{aligned} \tag{2}$$
The proof of this follows (independently) from the existence and regularity theory for elliptic equations in divergence form, so I shall delay the proof until we discuss elliptic equations in divergence form. (The rough idea is that (2) is the Dirichlet problem for an equation in divergence form, so we can solve it if $f, \varphi \in L^2(B)$ by the existence theory for such Dirichlet problems. Now if all the derivatives of f and φ are in $L^2(B)$, as is the case if f and φ are smooth, then all the derivatives of u are in L^2 by the regularity theory for equations in divergence form. But that means there exists a solution u in $C^\infty(\overline{B})$!) Now suppose we wanted to solve (2) for $f \in C^{0,\mu}(\overline{B})$ and $\varphi \in C^0(\partial B)$. By scaling, we can suppose $B = B_1(0)$. Recall that we can extend f to $f \in C^{0,\mu}_c(\mathbb{R}^n)$ with $|f|_{0,\mu;\mathbb{R}^n} \leq C|f|_{0,\mu;B}$ for some constant $C = C(n, \mu) \in (0, \infty)$. Note that we can extend φ to be a continuous function on $\mathbb{R}^n$ by letting $\varphi(x) = \chi(|x|)\varphi(x/|x|)$, where $\chi \in C^\infty([0, \infty))$ satisfies $\chi(r) = 0$ for $r \in [0, 1/4]$ and $\chi(r) = 1$ for $r \geq 1/2$. Using convolution, we can approximate f and φ by $f_k \in C^\infty(\overline{B})$ and $\varphi_k \in C^\infty(\overline{B})$ such that
$$f_k \to f \text{ and } \varphi_k \to \varphi \text{ uniformly on } \overline{B} \text{ as } k \to \infty$$
and
$$\sup_B |f_k| \leq \sup_B |f|, \quad [f_k]_{\mu;B} \leq [f]_{\mu;\mathbb{R}^n}, \quad \sup_B |\varphi_k| \leq \sup_B |\varphi|. \tag{3}$$
That is, let $\phi_1 \in C^\infty(\mathbb{R}^n)$ such that $\phi_1 = 0$ on $\mathbb{R}^n \setminus B_1(0)$, $\phi_1 \geq 0$, and $\int_{B_1(0)} \phi_1 = 1$. Let $\phi_\sigma(x) = \sigma^{-n}\phi_1(x/\sigma)$ for all $x \in \mathbb{R}^n$ and $\sigma > 0$. Note that $\phi_\sigma = 0$ on $\mathbb{R}^n \setminus B_\sigma(0)$, $\phi_\sigma \geq 0$, and $\int_{B_\sigma(0)} \phi_\sigma = 1$. For $\sigma_k \downarrow 0$, let
$$f_k(x) = \int_{\mathbb{R}^n} f(x-y)\phi_{\sigma_k}(y)dy \text{ and } \varphi_k(x) = \int_{\mathbb{R}^n} \varphi(x-y)\phi_{\sigma_k}(y)dy.$$

To see that $f_k \to f$ uniformly, let $\varepsilon > 0$. Since f is uniformly continuous on $\overline{B}$, there is a $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$. Thus for k large enough that $\sigma_k < \delta$ and for $x \in \overline{B}$,

$$|f_k(x) - f(x)| \leq \int_{B_\sigma(0)} |f(x-y) - f(y)| \phi_{\sigma_k}(y) dy < \int_{B_\sigma(0)} \varepsilon \phi_{\sigma_k}(y) dy = \varepsilon.$$

Similarly $\varphi_k \to \varphi$ uniformly in $\overline{B}$. We compute for $x \in \overline{B}$,

$$|f_k(x)| \leq \int_{\mathbb{R}^n} |f(x-y)| \phi_{\sigma_k}(y) dy \leq \sup_B |f| \int_{\mathbb{R}^n} \varepsilon \phi_{\sigma_k}(y) dy = \sup_B |f|$$

and similarly $\sup_B |\varphi_k| \leq \sup_B |\varphi|$. We also compute for $x, x' \in \overline{B}$,

$$|f_k(x) - f_k(x')| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x'-y)| \phi_{\sigma_k}(y) dy \leq [f]_{\mu;\mathbb{R}^n} |x-x'|^\mu \int_{\mathbb{R}^n} \phi_{\sigma_k}(y) dy = [f]_{\mu;\mathbb{R}^n} |x-x'|^\mu.$$

Now recall that since f_k and φ_k are smooth, there exists $u_k \in C^\infty(\overline{B})$ such that

$$\begin{aligned} \Delta u_k &= f_k \text{ in } B, \\ u_k &= \varphi_k \text{ on } \partial B. \end{aligned}$$

By the a priori estimates,

$$\sup_B |u_k - u_l| \leq \sup_{\partial B} |\varphi_k - \varphi_l| + C \sup_{\partial B} |f_k - f_l|$$

for some constant $C = C(n, \mu, L) \in (0, \infty)$. Since $\varphi_k \to \varphi$ and $f_k \to f$ uniformly in $\overline{B}$ as $k \to \infty$, u_k is Cauchy in $C^0(\overline{B})$ and thus u_k converges to some $u \in C^0(\overline{B})$ uniformly in $\overline{B}$. By the interior Schauder estimates and (3),

$$|u_k|_{2,\mu;B_\rho(0)} \leq C(|\varphi|_{0;B} + |f|_{0,\mu;B})$$

for some constant $C = C(n, \mu, L, \rho) \in (0, \infty)$. Thus, after passing to a subsequence, $u_k \to u$ in C^2 on compact sets in B and $u \in C^{2,\mu}(B)$. Therefore $u \in C^0(\overline{B}) \cap C^{2,\mu}(B)$ solves the Dirichlet problem (2).

Note that if additionally $\varphi \in C^{2,\mu}(\overline{B})$, then we can extend to $\varphi \in C^{2,\mu}_c(\mathbb{R}^n)$ with $|\varphi|_{2,\mu;\mathbb{R}^n} \leq C|\varphi|_{2,\mu;B}$ for some constant $C = C(n, \mu) \in (0, \infty)$ and for φ_k as defined above using convolution,

$$|\varphi_k|_{2,\mu;B} \leq |\varphi|_{2,\mu;\mathbb{R}^n}.$$

Thus by the global Schauder estimates,

$$|u_k|_{2,\mu;B} \leq C(|\varphi|_{2,\mu;B} + |f|_{0,\mu;B})$$

for some constant $C = C(n, \mu, L) \in (0, \infty)$. Thus, after passing to a subsequence, $u_k \to u$ in $C^2(\overline{B})$ and $u \in C^{2,\mu}(\overline{B})$.

3 Step 2: Solution for L implies solution for Δ

Theorem 1. *Let Ω be a bounded $C^{2,\mu}$ domain in $\mathbb{R}^n$. Let*

$$L = a^{ij} D_{ij} + b^i D_i + c$$

be an elliptic operator with coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$ and $c \leq 0$ in Ω. The Dirichlet problem

$$\begin{aligned} \Delta u &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{4}$$

has a (unique) solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^{2,\mu}(\overline{\Omega})$ if and only if the Dirichlet problem

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{5}$$

has a (unique) solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^{2,\mu}(\overline{\Omega})$

Remark 1. *Using an approximation argument similar to the argument in Section 2, we can show that the theorem holds when $\varphi \in C^0(\partial\Omega)$ instead of $\varphi \in C^{2,\mu}(\overline{\Omega})$. That is, (4) has a solution $u \in C^0(\overline{\Omega}) \cap C^{2,\mu}(\Omega)$ for every $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$ if and only if (5) has a solution $u \in C^0(\overline{\Omega}) \cap C^{2,\mu}(\Omega)$ for every $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$.*

Proof of Theorem 1. First observe that by letting $v = u - \varphi$, the Dirichlet problem $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ is equivalent to the Dirichlet problem $Lv = f - L\varphi$ in Ω and $v = 0$ on $\partial\Omega$. Thus it suffices to show that $\Delta u = f$ in Ω and $u = 0$ in $\partial\Omega$ has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$ if and only if $Lu = f$ in Ω and $u = 0$ in $\partial\Omega$ has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$.

Consider the one-parameter family of bounded linear operators L_t, $t \in [0, 1]$, given by

$$L_t = (1 - t)\Delta + tL : C_0^{2,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega}),$$

where $C_0^{2,\mu}(\overline{\Omega}) = \{u \in C^{2,\mu}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. To show that the Dirichlet problem

$$\begin{aligned} L_t u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{6}$$

has a (unique) solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$ is equivalent to showing the bounded linear operator L_t is invertible.

Suppose for some $s \in [0, 1]$, the Dirichlet problem

$$\begin{aligned} L_s u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

has a (unique) solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$. Equivalently, suppose L_s is invertible. We can rewrite (6) as

$$\begin{aligned} u &= L_s^{-1} f + L_s^{-1}(L_s - L_t)u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Thus solving (6) is equivalent to showing the map

$$T = L_s^{-1} f + L_s^{-1}(L_s - L_t) : C_0^{2,\mu}(\overline{\Omega}) \to C_0^{2,\mu}(\overline{\Omega})$$

has a fixed point. We will show this using the contraction mapping principle. Observe that for $u, v \in C_0^{2,\mu}(\overline{\Omega})$,

$$Tu - Tv = L_s^{-1}(L_s - L_t)(u - v) = (s - t)L_s^{-1}(L - \Delta)(u - v).$$

Hence

$$|Tu - Tv|_{2,\mu;\Omega} \leq |s - t| \|L_s^{-1}\| (\|L\| + \|\Delta\|) |u - v|_{2,\mu;\Omega}.$$

By the global Schauder estimates, $|u|_{2,\mu;\Omega} \leq C|L_s u|_{0,\mu;\Omega}$ for all $u \in C_0^{2,\mu}(\overline{\Omega})$ for some constant $C = C(n, \Omega, L) \in (0, \infty)$ independent of s, or equivalently $|L_s^{-1} f|_{2,\mu;\Omega} \leq C|f|_{0,\mu;\Omega}$ for all $f \in C^{0,\mu}(\overline{\Omega})$, or equivalently $\|L_s^{-1}\| \leq C$. Hence for $u, v \in C_0^{2,\mu}(\overline{\Omega})$,

$$|Tu - Tv|_{2,\mu;\Omega} \leq |s - t| C |u - v|_{2,\mu;\Omega}$$

for some constant $C = C(n, \Omega, L) \in (0, \infty)$ independent of s. Thus if $|s - t| < 1/2C$, T is a contraction mapping and thus has a unique fixed point. Therefore we conclude L_t is invertible if $|s - t| < 1/2C$.

Now partition the interval $[0, 1]$ into $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ where $t_j - t_{j-1} < 1/4C$ for all $j = 1, 2, \ldots, N$. If the Dirichlet Problem (4) has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$, then by the L_0 is invertible. By the discussion above, L_t is invertible for all $t \in [0, t_1]$, and thus L_t is invertible for all $t \in [t_1, t_2]$, and thus L_t is invertible for all $t \in [t_2, t_3]$,..., L_t is invertible for all $t \in [t_{N-1}, 1]$. That is, L_1 is invertible and thus the Dirichlet Problem (5) has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$. That is, if Dirichlet Problem (4) has unique solutions then Dirichlet Problem (5) has unique solutions. Similarly we can probe the converse that if Dirichlet Problem (5) has unique solutions then Dirichlet Problem (4) has unique solutions.

Note that there is an alternative way to finish the proof. Consider the set

$$S = \{t \in [0, 1] : L_t^{-1} \text{ exists}\}.$$

We have already shown using the Schauder estimates and contraction mapping principle that S is (relatively) open in $[0, 1]$. Using the Schauder estimates you can show that S is closed in $[0, 1]$. But $[0, 1]$ is connected, thus either $S = \emptyset$ or $S = [0, 1]$, i.e. L_t^{-1} exists for no $t \in [0, 1]$ or L_t^{-1} exists for all $t \in [0, 1]$. Hence the existence of Δ^{-1} is equivalent to the existence of L^{-1}, i.e. the Dirichlet Problem (4) has unique solutions if and only if Dirichlet Problem (5) has unique solutions. □

4 Subfunctions and superfunctions

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the differential equation

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega$$

where L is an elliptic operator with coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\Omega)$ for $\mu \in (0, 1)$ and $c \leq 0$ in Ω and $f \in C^{0,\mu}(\Omega)$. Recall the following definition:

Definition 1. *Let $u \in C^2(\Omega)$. u is a subsolution if $Lu \geq f$ in Ω. u is a supersolution if $Lu \leq f$ in Ω.*

We can generalize the concepts of subsolutions and supersolutions to functions in $C^0(\Omega)$ via the maximum principle:

Definition 2. *Let $u \in C^0(\Omega)$. u is a subsolution if for every ball $B \subset\subset \Omega$ and $w \in C^0(\overline{B}) \cap C^2(B)$ such that $Lw = f$ in B and $u \leq w$ on ∂B, $u \leq w$ in B. u is a supersolution if for every ball $B \subset\subset \Omega$ and $w \in C^0(\overline{B}) \cap C^2(B)$ such that $Lw = f$ in B and $w \leq u$ on ∂B, $w \leq u$ in B.*

We claim that if $u \in C^2(\Omega)$, Definitions 1 and 2 are equivalent. Let's check this for subsolutions, the case of supersolutions is similar. It is obvious by the maximum principle that if u is a subsolution in the sense of Definition 1, then u is a subsolution in the sense of Definition 2. Suppose u is a subsolution in the sense of Definition 2 and $Lu(y) < f(y)$ for some $y \in \Omega$. Then by continuity, for some $\delta > 0$, $Lu < f$ in $B_\delta(y)$. Let $w \in C^{2,\mu}(\overline{B_\delta(y)})$ be the solution to $Lw = f$ in $B_\delta(y)$ and $w = u$ on $\partial B_\delta(y)$. Then by the maximum principle, $w < u$ in $B_\delta(y)$, contradicting u being a subsolution by Definition 2. Since Definitions 1 and 2 are equivalent for functions in $C^2(\Omega)$, we will now simply talk about subsolutions and supersolutions.

Now we claim that subsolutions and supersolutions in the sense of Definition 2 satisfy a maximum principle:

Lemma 1. *Suppose $u, v \in C^0(\overline{\Omega})$ such that u is a subsolution and v is a supersolution (in the sense of Definition 2) and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. Let $M = \sup_\Omega (u - v) > 0$. Since $u - v \leq 0 < M$ on $\partial\Omega$, $v - w$ is not constant on Ω, so we can choose $x_0 \in \Omega$ and $R > 0$ such that $u(x_0) - v(x_0) = M$ and $u - v \not\equiv M$ on $\partial B_R(x_0)$. Let $\bar{u}, \bar{v} \in C^0(\overline{B_R(x_0)}) \cap C^2(B_R(x_0))$ such that $L\bar{u} = L\bar{v} = f$ in $B_R(x_0)$, $\bar{u} = u$ on $\partial B_R(x_0)$, and $\bar{v} = v$ on $\partial B_R(x_0)$. By the definition of subsolution and supersolution, $u \leq \bar{u}$ in $B_R(x_0)$ and $v \geq \bar{v}$ in $B_R(x_0)$. Thus, applying the maximum principle to the function $\bar{u} - \bar{v}$ such that $L(\bar{u} - \bar{v}) = 0$ in $B_R(x_0)$,

$$M = (u - v)(x_0) \leq (\bar{u} - \bar{v})(x_0) \leq \sup_{\partial B_R(x_0)} (\bar{u} - \bar{v}) = \sup_{\partial B_R(x_0)} (u - v) \leq M,$$

and thus we must have equality throughout. In particular $(\bar{u} - \bar{v})(x_0) = \sup_{\partial B_R(x_0)} (\bar{u} - \bar{v})$, i.e. $\bar{u} - \bar{v}$ has an interior maximum and thus by the strong maximum principle $\bar{u} - \bar{v} \equiv M$ in $B_R(x_0)$. Thus $u - v \equiv M$ on $\partial B_R(x_0)$, contrary to assumption. Therefore $u \leq v$ in Ω. □

It now follows by an identical proof using the maximum principle that we have the a priori estimates on subsolutions: if $u \in C^0(\overline{\Omega})$ is a subsolution, then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \sup_\Omega |f|$$

for some constant $C = C(n, L, \Omega) \in (0, \infty)$.

Lemma 2. *Suppose $u \in C^0(\overline{\Omega})$ is a subsolution and B be a ball such that $B \subset\subset \Omega$. We define the lift of u to be the function $U \in C^0(\overline{\Omega})$ defined by letting U on B be the solution in $C^0(\overline{B}) \cap C^2(B)$ to $LU = f$ in B and $U = u$ on ∂B and letting $U = u$ on $\Omega \setminus B$. The lift U is a subsolution.*

Proof. Let B' be a ball such that $B' \subset\subset \Omega$ and let $w \in C^0(\overline{B'}) \cap C^2(B')$ such that $Lw = f$ in B' and $U \leq w$ on $\partial B'$. We want to show that $U \leq w$ in B'. Since u is a subsolution and $u \leq U \leq w$ on $\partial B'$, $u \leq w$ in B'. In particular, since $U = u$ on $B' \setminus B$, $U \leq w$ on $B' \setminus B$. Since $LU = f$ in $B \cap B'$, $U \leq w$ on $B \cap \partial B'$ and $U = u \leq w$ on $\partial B \cap B'$, by the maximum principle $U \leq w$ in $B \cap B'$. □

Lemma 3. *Suppose $u, v \in C^0(\overline{\Omega})$ are both subsolutions. Then* $\max\{u, v\}$ *is a subsolution.*

Proof. Let B be a ball such that $B \subset\subset \Omega$ and let $w \in C^0(\overline{B}) \cap C^2(B)$ such that $Lw = f$ in B and $\max\{u, v\} \leq w$ on ∂B. Since u is a subsolution and $u \leq \max\{u, v\} \leq w$ on ∂B, $u \leq w$ in B. Similarly $v \leq w$ in B. Therefore $\max\{u, v\} \leq w$ in B. □

5 Step 3: Perron Method

Now we are go to apply subsolutions and supersolutions to solve the Dirichlet problem

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

where Ω, L, and f are as given above and $\varphi \in C^0(\partial\Omega)$.

Definition 3. *Let $u \in C^0(\Omega)$. u is a subfunction if u is a subsolution and $u \leq \varphi$ on $\partial\Omega$. u is a superfunction if u is a supersolution and $u \geq \varphi$ on $\partial\Omega$.*

Theorem 2. *Define $u : \overline{\Omega} \to \mathbb{R}$ by*

$$u(x) = \sup\{v(x) : v \in C^0(\overline{\Omega}) \text{ is a subfunction } \}$$

for all $x \in \overline{\Omega}$. Then $u \in C^2(\Omega)$ and $Lu = f$ in Ω.

Remark 2. *Note that there does indeed exist at least one subfunction. Assume without loss of generality that $\Omega \subset \{0 \leq x_1 \leq d\}$ for some $d > 0$. Recall the definition of λ as the positive function on Ω such that $a^{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. By the proof of the a priori estimates it is readily seen that*

$$v(x) = -\inf_{\partial\Omega} |\varphi| - (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f|}{\lambda} \tag{7}$$

is a subfunction provided $\alpha \geq 1$ is sufficiently large (depending on the coefficients of L). Moreover, from the a priori estimates the subfunctions are uniformly bounded above. Therefore u in the statement of Theorem 2 is well-defined and finite on Ω.

Remark 3. *Observe that Theorem 2 does not claim that $u = \varphi$ on $\partial\Omega$. However, if there is a solution $\widetilde{u} \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to the Dirichlet problem $L\widetilde{u} = f$ in Ω and $\widetilde{u} = \varphi$ on $\partial\Omega$, then $u = \widetilde{u}$. In particular, by the definition of u and the fact that $\widetilde{u}$ is a subfunction, $\widetilde{u} \leq u$ in Ω. Since $\widetilde{u}$ is also a superfunction, $v \leq \widetilde{u}$ in Ω for all subfunctions $v \in C^0(\overline{\Omega})$ by the maximum principle, so $u \leq \widetilde{u}$ in Ω.*

Proof of Theorem 2. Fix $x \in \Omega$ and let $R > 0$ such that $B_R(x) \subset\subset \Omega$. There is a sequence of subfunctions $v_j \in C^0(\overline{\Omega})$ such that $v_j(x) \to u(x)$ as $j \to \infty$. By replacing v_j with the max of v_j and the subfunction in (7), we may assume that v_j are uniformly bounded below. Let $V_j \in C^0(\overline{\Omega})$ be the lift of v_j on $B_R(x_0)$ so that $LV_j = f$ in $B_R(x)$, $V_j = v_j$ on $\partial B_R(x)$, and $V_j = v_j$ on $\Omega \setminus B_R(x)$. Recall that V_j is a subfunction and observe that $\lim_{j\to\infty} V_j(x) = u(x)$. By the lower bound on v_j, the a priori estimates, and the local Schauder estimates,

$$|V_j|_{2,\mu;B_{R/2}(x)} \leq C(\sup_{\partial\Omega} \varphi^+ + |f|_{0,\mu;\Omega})$$

for some constant $C = C(n, \mu, L) \in (0, \infty)$. Thus by Arzela-Ascoli, after passing to a subsequence, V_j converges in $C^2(B_{R/2}(x))$ to some function $V \in C^2(B_{R/2}(x))$ such that $LV = f$ in $B_{R/2}(x)$, $V \leq u$ in $B_{R/2}(x)$, and $V(x) = u(x)$.

We claim that $V = U$ in $B_{R/16}(x)$. Suppose $V(z) < U(z)$ for some $z \in B_{R/16}(x)$. By the definition of $U(z)$, there is a subfunction $w \in C^0(\overline{\Omega})$ such that $V(z) < w(z) \leq u(z)$. Let $w_j = \max\{V_j, w\}$. Let $W_j \in C^0(\overline{\Omega})$ be the lift of w_j on $B_{R/4}(z)$ so that $LW_j = f$ in $B_{R/4}(z)$, $W_j = w_j$ on $\partial B_{R/4}(z)$, and $W_j = w_j$ on $\Omega \setminus B_{R/4}(z)$. Observe that W_j is a subfunction and observe that $\lim_{j\to\infty} W_j(x) = u(x)$ and $V(z) < \lim_{j\to\infty} W_j(z) \leq u(z)$. By the lower bound on v_j, the a priori estimates, and the local Schauder estimates,

$$|W_j|_{2,\mu;B_{R/8}(z)} \leq C(\sup_{\partial\Omega} \varphi^+ + |f|_{0,\mu;\Omega})$$

for some constant $C = C(n, \mu, L) \in (0, \infty)$. Thus by Arzela-Ascoli, after passing to a subsequence, W_j converges in $C^2(B_{R/8}(z))$ to some function $W \in C^2(B_{R/8}(z))$ such that $LW = f$ in $B_{R/8}(z)$, $V \leq W \leq u$ in $B_{R/8}(z)$, $V(x) = W(x) = u(x)$, and $V(z) < W(x) \leq u(x)$. Since $LV = LW = f$ in $B_{R/8}(z)$ and $V(x) = W(x)$ at $x \in B_{R/8}(z)$, by the strong maximum principle $V = W$ in $B_{R/8}(z)$, contradicting the fact that $V(z) < W(z)$. Therefore $V = u$ in $B_{R/16}(x)$. Thus $u = V \in C^2(B_{R/16}(x))$ with $Lu(x) = LV(x) = f(x)$. □

6 Step 4: Boundary continuity and barriers

Definition 4. *Let Ω be a bounded domain and $\varphi \in C^0(\partial\Omega)$. A sequence of functions $\{w_i^+\}$ in $C^0(\overline{\Omega})$ is an upper barrier) at $\xi \in \partial\Omega$ if*

(i) w_i^+ is a superfunction relative to φ in Ω, i.e. w_i^+ is a supersolution and $\varphi \leq w_i^+$ on $\partial\Omega$, and

(ii) $w_i^+(\xi) \to \varphi(\xi)$ as $i \to \infty$.

A sequence of functions $\{w_i^-\}$ in $C^0(\overline{\Omega})$ is an lower barrier) in Ω relative to L, f, and φ at $\xi \in \partial\Omega$ if

(i) w_i^- is a subfunction relative to φ in Ω, i.e. w_i^+ is a subsolution and $w_i^- \leq \varphi$ on $\partial\Omega$, and

(ii) $w_i^-(\xi) \to \varphi(\xi)$ as $i \to \infty$.

Definition 5. *Let Ω be a bounded domain and $\varphi \in C^0(\partial\Omega)$. Let M^- and M^+ be real numbers such that*

$$M^- \leq u \leq M^+$$

for any solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *to the Dirichlet problem* $Lu = f$ *in* Ω *and* $u = \varphi$ *on* $\partial\Omega$*. For example, since* $c \leq 0$ *in* Ω *we can take*

$$M^+ = -M^- = \sup_{\partial\Omega} |\varphi| + C \sup_{\Omega} |f| \tag{8}$$

for an appropriate constant $C = C(n, L, \Omega) \in (0, \infty)$*. A sequence of functions* $\{w_i^+\}$ *in* $C^0(\overline{\Omega})$ *is an local upper barrier) at* $\xi \in \partial\Omega$ *if there is an open neighborhood* $\mathcal{N}$ *of* ξ *such that*

(i) w_i^+ *is a supersolution in* $\mathcal{N} \cap \Omega$*,*

(ii) $w_i^+ \geq \varphi$ *on* $\mathcal{N} \cap \partial\Omega$*,*

(iii) $w_i^+ \geq M^+$ *on* $\partial\mathcal{N} \cap \Omega$*, and*

(iv) $w_i^+(\xi) \to \varphi(\xi)$ *as* $i \to \infty$*.*

A sequence of functions $\{w_i^-\}$ *in* $C^0(\overline{\Omega})$ *is an local lower barrier) at* $\xi \in \partial\Omega$ *if there is an open neighborhood* $\mathcal{N}$ *of* ξ *such that*

(i) w_i^- *is a subsolution in* $\mathcal{N} \cap \Omega$*,*

(ii) $w_i^- \leq \varphi$ *on* $\mathcal{N} \cap \partial\Omega$*,*

(iii) $w_i^- \leq M^-$ *on* $\partial\mathcal{N} \cap \Omega$*, and*

(iv) $w_i^-(\xi) \to \varphi(\xi)$ *as* $i \to \infty$*.*

Note that if we can construct a function $m^+ \in C^{2,\mu}(\overline{\Omega})$ such that

(a) $Lm^+ \leq f$ in Ω,

(b) $m^+ \geq \varphi$ on $\partial\Omega$, and

(c) $m^+ \leq M^+$ on $\overline{\Omega}$,

then any local upper barrier $\{w_i^+\}$ can be extended to an upper barrier $\{\bar{w}_i^+\}$ on Ω defined by

$$\bar{w}_i^+(x) = \begin{cases} \min\{w_i^+, m^+\} & \text{in } \mathcal{N} \cap \Omega, \\ m^+ & \text{in } \Omega \setminus \mathcal{N}. \end{cases}$$

Observe that in the special case that $L = \Delta$ and $f = 0$ in Ω, we can choose $m^+(x) = M^+$ for all $x \in \Omega$. This doesn't quite work in general. But if Ω is a bounded domain with Ω and we choose M^+ by (8), we can construct such a function m^+ (exercise for the reader).

Theorem 3. *Let* u *be the solution to* $Lu = f$ *in* Ω *constructed in Lemma 2. If there exists local upper and lower barriers at* $\xi \in \partial\Omega$*, then* $u(x) \to \varphi(\xi)$ *as* $x \to \xi$*.*

Proof. Let w_i^+ be upper barriers at ξ and w_i^- be lower barriers at ξ, which exist since local upper and lower barriers exist. By the maximum principle,

$$w_i^- \leq u \leq w_i^+ \text{ in } \Omega. \tag{9}$$

For every $\varepsilon > 0$, for i sufficiently large,

$$\varphi(\xi) - \varepsilon < w_i^-(\xi) \text{ and } w_i^+(\xi) \leq \varphi(\xi) + \varepsilon.$$

By continuity, for some $\delta > 0$ (depending on ε and i),

$$\varphi(\xi) - \varepsilon < w_i^-(x) \text{ and } w_i^+(\xi) \leq \varphi(\xi) + \varepsilon \text{ for } x \in \Omega \cap B_\delta(\xi). \tag{10}$$

Thus by (9) and (10),

$$\varphi(\xi) - \varepsilon \leq u(x) \leq \varphi(\xi) + \varepsilon \text{ for } x \in \Omega \cap B_\delta(\xi).$$

Therefore $u(x) \to \varphi(\xi)$ as $x \to \xi$. □

Consider our case where $L = a^{ij} D_{ij} + b^i D_i + c$ is an elliptic operator with coefficients $a^{ij}, b^i, c \in C^0(\overline{\Omega})$ and $c \leq 0$ in Ω and f is a bounded function on Ω. Then the upper and lower barriers are determined by a single function $w \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, simply called a *barrier*, such that

(a) $Lw \leq -1$ in Ω and

(b) $w > 0$ on $\partial\Omega \setminus \{\xi\}$ and $w(\xi) = 0$.

Now for $\varepsilon_i \downarrow 0$, let

$$w_i^+ = \varphi(\xi) + \varepsilon_i + k_i w, \quad w_i^- = \varphi(\xi) - \varepsilon_i - k_i w$$

on Ω for some $k_i > 0$. Now if $k_i \geq \sup_\Omega |f - c\varphi(\xi)|$,

$$Lw_i^+ = c\varphi(\xi) + c\varepsilon_i + k_i Lw \leq c\varphi(\xi) + 0 - k_i \leq f$$

in Ω and similarly $Lw_i^- \geq f$ in Ω. For some $\delta_i > 0$, $|\varphi(x) - \varphi(\xi)| < \varepsilon_i$ for all $x \in \partial\Omega \cap B_{\delta_i}(\xi)$, so

$$w_i^- \leq \varphi \leq w_i^+ \text{ on } \partial\Omega \cap B_{\delta_i}(\xi).$$

Since $w > 0$ on $\partial\Omega \setminus B_{\delta_i}(\xi)$, we can choose k_i large enough that

$$w_i^- \leq \varphi \leq w_i^+ \text{ on } \partial\Omega.$$

Finally note that since $\varepsilon_i \downarrow 0$ and $w(\xi) = 0$, $w_i^+(\xi) \to \varphi(\xi)$ and $w_i^-(\xi) \to \varphi(\xi)$ as $i \to \infty$. Therefore $\{w_i^+\}$ are upper barriers and $\{w_i^-\}$ are lower barriers. Note that we could also determine local upper and lower barriers by a single function $w \in C^0(\overline{\Omega} \cap \mathcal{N}) \cap C^2(\Omega \cap \mathcal{N})$ for some open neighborhood $\mathcal{N}$ of ξ, where w is simply called a *local barrier*, by

(a) $Lw \leq -1$ in $\Omega \cap \mathcal{N}$ and

(b) $w > 0$ on $\partial(\Omega \cap \mathcal{N}) \setminus \{\xi\}$ and $w(\xi) = 0$.

Again for $\varepsilon_i \downarrow 0$, we let

$$w_i^+ = \varphi(\xi) + \varepsilon_i + k_i w, \quad w_i^- = \varphi(\xi) - \varepsilon_i - k_i w$$

on $\Omega \cap \mathcal{N}$ for some $k_i > 0$. We can show as before that $Lw_i^- \geq f \geq Lw_i^+$ in $\Omega \cap \mathcal{N}$ and $w_i^- \leq \varphi \leq w_i^+$ on $\mathcal{N} \cap \partial\Omega$ if k_i is chosen to be sufficiently large and that $w_i^{\pm}(\xi) \to \varphi(\xi)$ as $i \to \infty$. Clearly if we choose k_i sufficiently large, then $w_i^- \leq M^-$ and $w_i^+ \geq M^+$ on $\partial\mathcal{N} \cap \Omega$.

We claim that if Ω satisfies the exterior sphere condition at ξ, i.e. there is a ball $B_R(y)$ such that $\overline{B_R(y)} \cap \overline{\Omega} = \{\xi\}$, then we can construct such a local barrier w. Let

$$w(x) = \tau(R^{-\sigma} - |x-y|^{-\sigma})$$

on $\mathbb{R}^n$ for $\tau, \sigma > 0$ to be determined. We compute

$$L(R^{-\sigma} - |x-y|^{-\sigma}) \leq |x-y|^{-\sigma-2} \left(-\sigma(\sigma+2) \sum_{i,j=1}^{n} \frac{a^{ij}(x_i - y_i)(x_j - y_j)}{|x-y|^2} + \sigma \sum_{i=1}^{n} (a^{ii} + b^i(x_i - y_i)) \right)$$

which is negative and bounded away from zero provided σ is sufficiently large depending on n, L, R, and τ. Thus for σ and τ sufficiently large, $Lw \leq -1$. Therefore w is a barrier. Thus we have shown

Theorem 4. *Suppose Ω satisfies the exterior sphere condition (for example if Ω is a C^2 domain). Let $L = a^{ij}D_{ij} + b^i D_i + c$ be strictly elliptic with bounded coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\Omega)$ and $c \leq 0$. Let f be bounded and in $C^{0,\mu}(\Omega)$ and $\varphi \in C^0(\partial\Omega)$. Then the Dirichlet problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

has a unique solution $u \in C^0(\overline{\Omega}) \cap C^{2,\mu}(\Omega)$.

7 Solutions in $C^{2,\mu}(\overline{\Omega})$

Theorem 5. *Suppose Ω is a $C^{2,\mu}$ domain. Let $L = a^{ij}D_{ij} + b^i D_i + c$ be strictly elliptic with coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$ and $c \leq 0$. Let $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^{2,\mu}(\overline{\Omega})$. Then the Dirichlet problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$.

Proof. Recall that the Dirichlet problem has a unique solution $u \in C^0(\overline{\Omega}) \cap C^{2,\mu}(\Omega)$. It remains to show that u is in $C^{2,\mu}$ up to the boundary of $\overline{\Omega}$. In fact, we can show that for every point $\xi \in \partial\Omega$, there is an open neighborhood D of ξ in $\mathbb{R}^n$ such that $u \in C^{2,\mu}(D \cap \overline{\Omega})$. Since Ω is a $C^{2,\mu}$ domain, there is a $C^{2,\mu}$ diffeomorphism $\Psi : D \cap \overline{\Omega} \to \overline{B_1(0)}$ for some neighborhood D of ξ such that

$$\Psi(D \cap \partial\Omega) = T \subset \partial B_1(0).$$

Under the transformation Ψ, $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ transforms to

$$\begin{aligned}\widetilde{L}\widetilde{u} &= \widetilde{f} \text{ in } B_1(0),\\ \widetilde{u} &= \widetilde{\varphi} \text{ on } T,\end{aligned}$$

where $\widetilde{u} = u \circ \Psi^{-1}$, $\widetilde{L}$ is strictly elliptic, the coefficients of $\widetilde{L}$ are in $C^{0,\mu}(\overline{B_1(0)})$, $\widetilde{f} = f \circ \Psi^{-1} \in C^{0,\mu}(\overline{B_1(0)})$, and $\widetilde{\varphi} = \varphi \circ \Psi^{-1} \in C^{2,\mu}(\overline{B_1(0)})$. Note that $w = \widetilde{u}$ is a solution to

$$\begin{aligned}\widetilde{L}v &= \widetilde{f} \text{ in } B_1(0),\\ w &= \widetilde{u} \text{ on } \partial B_1(0).\end{aligned} \tag{11}$$

Let $\rho > 0$ such that $\partial B_1(0) \cap B_\rho(\xi) \subset T$. Consider $\chi = \widetilde{u}|_{\partial B_1(0)} \in C^0(\partial B_1(0)) \cap C^{2,\mu}(T)$. Let $\chi_j \in C^\infty(\overline{B_1(0)})$ such that $\chi_j \to \widetilde{u}$ uniformly on $\partial B_1(0)$, $|\chi_j|_{2,\mu,\partial B_1(0)} \leq C|\widetilde{u}|_{2,\mu,\partial B_1(0)}$, and $|\chi_j|_{2,\mu,\partial B_1(0)\cap B_\rho(y)} \leq C|\widetilde{u}|_{2,\mu,\partial B_1(0)\cap B_\rho(y)}$ for some constant $C = C(n,\mu) \in (0,\infty)$. By the solution to the Dirichlet problem on a ball, there exists solutions $u_j \in C^{2,\mu}(\overline{B_1(0)})$ to

$$\begin{aligned}\widetilde{L}u_j &= \widetilde{f} \text{ in } B_1(0),\\ u_j &= \chi_j \text{ on } \partial B_1(0).\end{aligned}$$

We claim that by the maximum principle, interior Schauder estimates, and Schauder estimates at the boundary near ξ, u_j converges to some function v uniformly on $\overline{B_1(0)}$, in C^2 on compact subsets of $B_1(0)$, and in $C^2(\overline{\Omega \cap B_{\rho/2}(\xi)})$ such that $v \in C^0(\overline{B_1(0)}) \cap C^{2\,mu}(B_1(0)) \cap C^{2\,mu}(\overline{B_1(0) \cap B_{\rho/2}(y)})$ and $w = v$ solves (11). By the maximum principle,

$$\sup_{B_1(0)} |v_j - v_k| \leq \sup_{\partial B_1(0)} |\chi_j - \chi_k|,$$

so since $\chi_j \to \widetilde{u}$ uniformly on $\partial B_1(0)$, v_j is Cauchy in $C^0(\overline{B_1(0)})$ and converges to some $v \in C^0(\overline{B_1(0)})$ as $j \to \infty$ with $v = \widetilde{u}$ on $\partial B_1(0)$. By the interior Schauder estimates and the a priori estimates,

$$\begin{aligned}|v_j|_{2,\mu;B_\rho(0)} &\leq C(|\chi_j|_{0;\partial B_1(0)} + |\widetilde{f}|_{0,\mu;B_1(0)})\\ &\leq C(|\widetilde{u}|_{0;\partial B_1(0)} + |\widetilde{f}|_{0,\mu;B_1(0)})\end{aligned}$$

for some constant $C = C(n,\mu,\widetilde{L},\rho) \in (0,\infty)$, so after passing to a subsequence $v_j \to v$ in C^2 on compact subsets of $B_1(0)$ and $v \in C^2(B_1(0))$ with $Lv = \widetilde{f}$ in $B_1(0)$. By the local Schauder estimates at the boundary and the a priori estimates,

$$\begin{aligned}|v_j|_{2,\mu;B_1(0)\cap B_{\rho/2}(\xi)} &\leq C(|\chi_j|_{0;\partial B_1(0)} + |\widetilde{f}|_{0,\mu;B_1(0)} + |\chi_j|_{2,\mu;\partial B_1(0)\cap B_\rho(\xi)})\\ &\leq C(|\widetilde{u}|_{0;\partial B_1(0)} + |\widetilde{f}|_{0,\mu;B_1(0)} + |\widetilde{\varphi}|_{2,\mu;\partial B_1(0)\cap B_\rho(\xi)})\end{aligned}$$

for some constant $C = C(n,\mu,\widetilde{L},\rho) \in (0,\infty)$, so after passing to a subsequence $v_j \to v$ in $C^2(\overline{B_1(0) \cap B_{\rho/2}(\xi)})$ and $v \in C^{2,\mu}(\overline{B_1(0) \cap B_{\rho/2}(\xi)})$. Clearly,

$$\begin{aligned}\widetilde{L}v &= \widetilde{f} \text{ in } B_1(0),\\ v &= \widetilde{u} \text{ on } \partial B_1(0),\end{aligned}$$

but by the uniqueness of solutions to the Dirichlet problem $v \equiv \widetilde{u}$ on $\overline{B_1(0)}$. Therefore $\widetilde{u} \in C^{2,\mu}(B_1(0) \cup T)$. Thus $u \in C^{2,\mu}(D \cap \overline{\Omega})$ as claimed and it follows that $u \in C^{2,\mu}(\overline{\Omega})$.

Then by the uniqueness of solutions to the Dirichlet problem this will imply that $\widetilde{u} = v$ and thus $\widetilde{u} \in C^{2,\mu}(\overline{B_1(0) \cap B_{\rho/2}(\xi)})$. It follows that u is $C^{2,\mu}$ up to the boundary of Ω in a neighborhood of ξ. Since ξ is arbitrary, $u \in C^{2,\mu}(\overline{\Omega})$. □

8 Fredholm alternative

Theorem 6 (Fredholm alternative)**.** *Suppose Ω is a $C^{2,\mu}$ domain. Let $L = a^{ij}D_{ij} + b^i D_i + c$ be strictly elliptic with coefficients $a^{ij}, b^i, c \in C^{0,\mu}(\overline{\Omega})$. Either*

(a) the homogeneous problem, $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$, has only the trivial solution, in which case the inhomogeneous problem, $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$, has a unique solution in $u \in C^{2,\mu}(\overline{\Omega})$ for all $f \in C^{0,\mu}(\overline{\Omega})$ and $\varphi \in C^{2,\mu}(\overline{\Omega})$, or

(b) the space of solutions to the homogeneous problem forms a nontrivial finite-dimensional subspace of $C^{2,\mu}(\overline{\Omega})$.

Proof. We can suppose $\varphi = 0$ since the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ is equivalent to the Dirichlet problem $Lv = f - L\varphi$ in Ω, $v = 0$ on $\partial\Omega$, where $v = u - \varphi$. For $\sigma \geq \sup_\Omega c$, let

$$L_\sigma = L - \sigma : C_0^{20,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega}),$$

where

$$C_0^{20,\mu}(\overline{\Omega}) = \{u \in C^{2,\mu}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}.$$

We have shown that the Dirichlet problem $L_\sigma u = f$ in Ω, $u = 0$ on $\partial\Omega$ has a unique solution u and thus the inverse map $L_\sigma^{-1} : C^{0,\mu}(\overline{\Omega}) \to \mathcal{B}$ exists. Moveover, by the global Schauder estimates and a priori estimates (Corollary 2 in Lecture 7), L_σ^{-1} is a bounded linear map. By Arzela-Ascoli,

$$L_\sigma^{-1} : C^{0,\mu}(\overline{\Omega}) \to C_0^{20,\mu}(\overline{\Omega}) \hookrightarrow C^{0,\mu}(\overline{\Omega})$$

is a compact mapping. We know $Lu = f$ in Ω is equivalent to

$$u + \sigma L_\sigma^{-1} u = L_\sigma^{-1} f \text{ in } \Omega.$$

Note in particular that if $u \in C^{0,\mu}(\overline{\Omega})$ satisfies $u + \sigma L_\sigma^{-1} u = L_\sigma^{-1} f$, then $u = L_\sigma^{-1}(f - \sigma u) \in C_0^{2,\mu}(\overline{\Omega})$ so $Lu = f$ in Ω makes sense and holds true. Ny standard functional analysis regarding compact operators, $(1 + \sigma L_\sigma^{-1})$ is either invertible, in which case we get (a), or $(1 + \sigma L_\sigma^{-1})$ has a nontrivial finite-dimensional kernel, in which case we get (b). □

References: Gilbarg and Trudinger, Section 6.3 (Dirichlet problem in general) and Section 2.8 (Perron method and barriers for harmonic functions).